International Journal of Theoretical Physics, Vol. 38, No. 2, 1999

Strings and the Gauge Theory of Spacetime Defects

Yishi Duan[1] **and Ying Jiang**[1,2]

Received May 27, 1998

We present a new topological invariant to describe space–time defects which is closely related to the torsion tensor in a Riemann–Cartan manifold. By virtue of the topological current theory and ϕ-mapping method, we show that there must exist multistring objects generated from the zero points of the ϕ-mapping. These strings are topologically quantized. The topological quantum numbers are the winding numbers described by the Hopf indices and the Brouwer degrees of the ϕ-mapping.

1. INTRODUCTION

The landscape of fundamental physics has changed substantially in recent decades (Rovelli, 1998). On one hand, at the microscopic level the strong and weak interactions dominate, while the gravitational interaction is the weakest and seems not to play any role. On the other hand, all known interactions but gravitation, that is, the strong, weak, and electromagnetic interactions, are well described within the framework of relativistic quantum field theory in flat Minkowski space–time. So at first sight it seems that gravitation has no effects when we are concerned with elementary particle physics. But we know that this is not true (De Sabbata, 1994): in fact, if we consider the quantum theory in curved instead of flat Minkowski spacetime, we have some very important effects (for instance, neutron interferometry; De Sabbata *et al.*, 1991). Moreover, when we go to the microphysical level, that is, when we are concerned with elementary particle physics, we realize that the role of gravitation becomes very important and necessary and this happens in the first place when we consider the early universe or the Planck era. In this unprecedented state of affairs, a large number of theoretical

[1] Institute of Theoretical Physics, Lanzhou University, Lanzhou 730000, China; e-mail: ysduan@lzu.edu.cn, itp3@lzu.edu.cn.
[2] To whom correspondence should be addressed.

physicists from different backgrounds have begun to address the piece of the puzzle which is clearly missing: combining the two halves of the picture and understanding the quantum properties of the gravitational field—equivalently, understanding the quantum properties of spacetime.

An exciting outcome of the interplay between particle physics and cosmology is string theory (Hindmarsh and Kibble, 1995). Strings are linear defects, analogous to those topological defects found in some condensed matter systems such as vortex lines in liquid helium, flux tubes in type II superconductors, or disclination lines in liquid crystals, and they are closely related to the torsion tensor of the Riemann–Cartan manifold (Duan *et al.*, 1994). String theory is strongly believed to solve the short-distance problems of quantum gravity at the Planck scale by providing a fundamental length $l_{\rm str} = \sqrt{\hbar c/T}$, where T is the string tension, and provides a bridge between the physics of the very small and the very large.

In recent years, string theory has reached an exciting stage, where models of various types [such as the Wess–Zumino–Witten model (Bakas, 1993, 1994), Ramond–Ramond charges of type II string theory (Cvetic and Youm, 1996), and the supersymmetric $SO(10)$ model (Jeannerot, 1996)] have been of much interest in differential geometry (Bakas and Sfetsos, 1996), field theory (Robinson and Ziabicki, 1996), and general relativity (Larsen and Sánchez, 1996). Though all these features make string theory very attractive, since most of them are based on concrete models, they are not very perfect and the topological properties of strings are not very clear.

As is well known, torsion is a slight modification of the Einstein theory of relativity [proposed in 1922–23 by Cartan (1922)], but is a generalization that appears to be necessary when one tries to reconcile general relativity with quantum theory. The main purpose of this paper is to establish a topological theory for a string through the ϕ-mapping method (Duan and Meng, 1993) and the theory of composed gauge potential (Duan and Lee, 1995) in a 4-dimensional Riemann–Cartan manifold X without any concrete models in the early universe. This theoretical framework includes three basic aspects: the generation of multistrings in a 4-dimensional Riemann–Cartan manifold, the topological charges of the multistrings, and their evolution equations.

This paper is organized as follows: In Section 2, we introduce a new topological invariant to describe spacetime defects, and using the decomposition of the gauge potential, we get the inner structure of the torsion. In Section 3, by means of the topological tensor current theory and the ϕ-mapping method, the multistrings are generated naturally at the zero points of the vector total field ϕ, and the topological quantum numbers of the length of these strings are just the Hopf indices and the Brouwer degrees of the ϕ-mapping. Furthermore, using some important relations, we obtain the Lagrangian density of multistrings and deduce the corresponding evolution equations in Section 4, and point out that the Lagrangian density is a generalization of that of Nielsen for strings and the evolution equations relate to the harmonic mapping in general relativity.

2. NEW TOPOLOGICAL INVARIANT AND SPACE–TIME DEFECTS

Using vierbein theory and the gauge potential decomposition, we will construct an invariant formulation of space–time defects. Space–time defects have been discussed from different points of view by many physicists. We will follow Duan and Zhang (1990), who studied space–time defects from the point of view of gauge field theory. The dislocation is described by the torsion

$$T^A_{\mu\nu} = D_\mu e^A_\nu - D_\nu e^A_\mu, \qquad \mu, \nu, A = 1, 2, 3, 4 \tag{1}$$

where e^A_μ is the vierbein field, and its gauge covariant derivative

$$D_\mu e^A_\nu = \partial_\mu e^A_\nu - \omega^{AB}_\mu(x) e^B_\nu$$

where ω^{AB}_μ stands for the spin connection of the Lorentz group.

By analogy with ‘t Hooft’s (1974) viewpoint, to establish a physical observable theory of space–time defects we must first define a gauge-invariant antisymmetric second-order tensor from the torsion tensor with respect to a unit vector field $N^A(x)$ as follows:

$$T_{\mu\nu} = T^A_{\mu\nu} N^A + e^A_\nu D_\mu N^A - e^A_\mu D_\nu N^A$$

By making use of

$$D_\mu N^A = \partial_\mu N^A - \omega^{AB}_\mu N^B$$

and (1), we can rewrite this as

$$T_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2}$$

where $A_\mu = e^A_\mu N^A$ is a kind of $U(1)$ gauge potential. This shows that the antisymmetric tensor $T_{\mu\nu}$ expressed in terms of A_μ is just like the $U(1)$ gauge field strength [i.e., the curvature on a $U(1)$ principle bundle with base manifold X], which is invariant for the $U(1)$-like gauge transformation

$$A'_\mu(x) = A_\mu(x) + \partial_\mu \Lambda(x) \tag{3}$$

where $\Lambda(x)$ is an arbitrary function.

Now, let us investigate the integral of the two-form $T = \frac{1}{2} T_{\mu\nu} dx^\mu \wedge dx^\nu$. It will be shown that, in topology, this is associated with (but is not!) the first Chern class, i.e.,

$$l = \int \frac{1}{2} T_{\mu\nu} dx^{\mu} \wedge dx^{\nu}$$

Since the integral quantity l carries neither the coordinate index nor the group index, it must be pointed that l is invariant under a general coordinate transformation as well as a local Lorentz transformation. Furthermore, in l there is another $U(1)$-like gauge invariance for (3). In fact, l is a new topological invariant and is used to measure the size of the space–time defects with the dimension of length.

Very commonly, the topological property of a physical system is much more important and worth investigating. It is our conviction that, in order to get a topological result, one should input the topological information from the beginning. Two useful tools—the ϕ-mapping method and composed gauge potential theory—do this. As A_μ is a kind of $U(1)$ gauge potential, for a section $\Phi(x)$ of the complex line bundle $L(X)$ with the base manifold X, which is looked upon as the order parameter of the spacetime defects, the corresponding $U(1)$-covariant derivative of $\Phi(x)$ with A_μ is

$$D_\mu \Phi(x) = \partial_\mu \Phi(x) - i \frac{2\pi}{L_p} A_\mu \Phi(x)$$

$$D_\mu \Phi^*(x) = \partial_\mu \Phi^*(x) + i \frac{2\pi}{L_p} A_\mu \Phi^*(x)$$

where L_p is the Planck length introduced to make both sides of the formula have the same dimension (Duan *et al.*, 1994). From the above equations, $A_\mu(x)$ can be expressed by

$$A_\mu(x) = \frac{iL_p}{4\pi\Phi\Phi^*} [(\Phi\partial_\mu\Phi^* - \Phi^*\partial_\mu\Phi) - (\Phi D_\mu\Phi^* - \Phi^* D_\mu\Phi)] \quad (4)$$

Further calculation shows that

$$A_\mu(x) = \frac{iL_p}{4\pi} \left(\frac{\Phi}{\sqrt{\Phi\Phi^*}} \partial_\mu \frac{\Phi^*}{\sqrt{\Phi\Phi^*}} - \frac{\Phi^*}{\sqrt{\Phi\Phi^*}} \partial_\mu \frac{\Phi}{\sqrt{\Phi\Phi^*}} \right) - \frac{iL_p}{4\pi\Phi\Phi^*} (\Phi D_\mu\Phi^* - \Phi^* D_\mu\Phi) \quad (5)$$

From the Chern–Weil homomorphism (Nash and Sen, 1983), we know that our new topological invariant is independent of the gauge potential, i.e., the last term in the RHS of equation (5) has nothing to do with the topological property in our theory. So we have a choice of many gauge potentials and the choice depends on the convenience of calculation. In the present work, we select A_μ as

$$A_\mu(x) = \frac{iL_p}{4\pi}\left(\frac{\Phi}{\sqrt{\Phi\Phi^*}}\partial_\mu\frac{\Phi^*}{\sqrt{\Phi\Phi^*}} - \frac{\Phi^*}{\sqrt{\Phi\Phi^*}}\partial_\mu\frac{\Phi}{\sqrt{\Phi\Phi^*}}\right)$$

satisfying relation (3) for the corresponding $U(1)$ gauge transformation $\Phi'(x) = \Lambda(x)\Phi(x)$. The section $\Phi(x)$ of the complex line bundle can be expressed by

$$\Phi(x) = \phi^1(x) + i\phi^2(x)$$

i.e., the section of the complex line bundle is equivalent to a 2-dimensional real vector field $\vec{\phi} = (\phi^1, \phi^2)$, and $\sqrt{\Phi\Phi^*} = \|\phi\| = \sqrt{\phi^a\phi^a}$ $(a = 1, 2)$. By defining the direction of the vector field $\vec{\phi}$ as

$$n^a(x) = \frac{\phi^a(x)}{\|\phi(x)\|} \tag{6}$$

we can obtain the expression of $A_\mu(x)$ in terms of n^a from (5),

$$A_\mu(x) = \frac{L_p}{2\pi}\varepsilon_{ab}n^a(x)\partial_\mu n^b(x) \tag{7}$$

Obviously, $n^a(x)n^a(x) = 1$, and $n^a(x)$ is a section of the sphere bundle $S(X)$ (Duan and Meng, 1993). The zero points of $\phi^a(x)$ are just the singular points of $n^a(x)$. Thus we get the total decomposition of the $U(1)$ gauge potential A_μ with the unit 2-vector field n^a, and because of the topological property of n^a, we input the topological information successfully.

3. SECOND-ORDER TOPOLOGICAL TENSOR CURRENT AND THE GENERATION OF STRINGS ON A RIEMANN–CARTAN MANIFOLD

In recent years, the topological current theory proposed by Duan has had a significant role in particle physics and field theory (especially gauge theory) (Duan and Meng, 1993; Duan and Lee, 1995; Duan and Zhang, 1990). The topological current theory can only be used to discuss the motion of pointlike particles (or pointlike singularities). In order to study string theory, we need to extend the concept, and introduce a second-order topological tensor current from the torsion.

From the above discussions, we can define a dual tensor $j^{\mu\nu}$ of $T_{\lambda\rho}$ as follows:

$$\begin{aligned} j^{\mu\nu} &= \frac{1}{2}\frac{1}{\sqrt{g_x}}\varepsilon^{\mu\nu\lambda\rho}T_{\lambda\rho} \\ &= \frac{1}{2}\frac{1}{\sqrt{g_x}}\varepsilon^{\mu\nu\lambda\rho}(\partial_\lambda A_\rho - \partial_\rho A_\lambda) \end{aligned} \tag{8}$$

With the decomposition of A_μ in (7), $j^{\mu\nu}$ can be expressed in terms of n^a by

$$j^{\mu\nu} = \frac{L_p}{2\pi}\frac{1}{\sqrt{g_x}}\varepsilon^{\mu\nu\lambda\rho}\varepsilon_{ab}\partial_\lambda n^a \partial_\rho n^b \tag{9}$$

which shows that $j^{\mu\nu}$ is just a second-order topological tensor current satisfying

$$j^{\mu\nu} = -j^{\nu\mu}, \qquad \frac{1}{\sqrt{g_x}}\partial_\mu(\sqrt{g_x}j^{\mu\nu}) = 0$$

i.e., $j^{\mu\nu}$ is antisymmetric and identically conserved.

Using (9) and

$$\partial_\mu n^a = \frac{1}{\|\phi\|}\partial_\mu\phi^a + \phi^a\partial_\mu\left(\frac{1}{\|\phi\|}\right), \qquad \frac{\partial}{\partial\phi^a}(\ln\|\phi\|) = \frac{\phi^a}{\|\phi\|^2}$$

which should be looked upon as generalized functions, we can express $j^{\mu\nu}$ by

$$j^{\mu\nu} = \frac{L_p}{2\pi}\frac{1}{\sqrt{g_x}}\varepsilon^{\mu\nu\lambda\rho}\varepsilon_{ab}\frac{\partial}{\partial\phi^c}\frac{\partial}{\partial\phi^a}(\ln\|\phi\|)\partial_\lambda\phi^c\partial_\rho\phi^b \tag{10}$$

By defining the general Jacobian determinants $J^{\mu\nu}(\phi/x)$ as

$$\varepsilon^{ab}J^{\mu\nu}(\phi/x) = \varepsilon^{\mu\nu\lambda\rho}\partial_\lambda\phi^a\partial_\rho\phi^b \tag{11}$$

and making use of the Laplacian relation in ϕ-space

$$\partial_a\partial_a \ln\|\phi\| = 2\pi\delta(\vec{\phi}), \qquad \partial_a = \frac{\partial}{\partial\phi^a}$$

we obtain the δ-like topological tensor current rigorously:

$$j^{\mu\nu} = \frac{1}{\sqrt{g_x}}L_p\delta(\vec{\phi})J^{\mu\nu}\left(\frac{\phi}{x}\right) \tag{12}$$

It is obvious that $j^{\mu\nu}$ is nonzero only when $\vec{\phi} = 0$.

Suppose that for the system of equations

$$\phi^1(x) = 0, \qquad \phi^2(x) = 0$$

there are l different solutions; when the solutions are regular solutions of $\vec{\phi}$ at which the rank of the Jacobian matrix $[\partial_\mu\phi^a]$ is 2, the solutions of $\vec{\phi}(x) = 0$ can be expressed in a parametrized way by

$$x^\mu = z_i^\mu(u^1, u^2), \qquad i = 1, \ldots, l \tag{13}$$

where the subscript i represents the ith solution and the parameters u^I ($I = 1, 2$) span a 2-dimensional submanifold with the metric tensor

$$g_{IJ} = g_{\mu\nu} \frac{\partial x^\mu}{\partial u^I} \frac{\partial x^\nu}{\partial u^J}$$

which is called the ith singular submanifold N_i in X. For each N_i, we can define a normal submanifold M_i in X which is spanned by the parameters v^A ($A = 1, 2$) with the metric tensor

$$g_{AB} = g_{\mu\nu} \frac{\partial x^\mu}{\partial v^A} \frac{\partial x^\nu}{\partial v^B}$$

and the intersection point of M_i and N_i is denoted by p_i. By virtue of the implicit function theorem, at the regular point p_i, the Jacobian matrix $J(\phi/v)$ satisfies

$$J\left(\frac{\phi}{v}\right) = \frac{D(\phi^1, \phi^2)}{D(v^1, v^2)} \neq 0 \tag{14}$$

The δ function on a submanifold $N_i \delta(N_i)$ satisfies the surface area relation (Schouten, 1951)

$$\int \delta(N_i) \sqrt{g_x}\, d^4x = \int_{N_i} \sqrt{g_u}\, d^2u$$

where $\sqrt{g_x} d^4x$ and $\sqrt{g_u} d^2u$ [$g_u = \det(g_{IJ})$] are invariant volume elements of X and N_i, respectively, and the expression for $\delta(N_i)$ is

$$\delta(N_i) = \int_{N_i} \frac{1}{\sqrt{g_x}} \delta^4(\vec{x} - \vec{z}_i(u^1, u^2)) \sqrt{g_u}\, d^2u$$

Following this, by analogy with the procedure for deducing $\delta(f(x))$, since

$$\delta(\vec{\phi}) = \begin{cases} +\infty & \text{for} \quad \vec{\phi}(x) = 0 \\ 0 & \text{for} \quad \vec{\phi}(x) \neq 0 \end{cases} = \begin{cases} +\infty & \text{for} \quad x \in N_i \\ 0 & \text{for} \quad x \notin N_i \end{cases} \tag{15}$$

we can expand the δ-function $\delta(\vec{\phi})$ as

$$\delta(\vec{\phi}) = \sum_{i=1}^{l} c_i \delta(N_i) \tag{16}$$

where the coefficients c_i must be positive, i.e., $c_i = |c_i|$.

In the following, we will discuss the dynamic form of the tensor current $j^{\mu\nu}$ and study the topological quantization of strings through the winding numbers (Guillemin and Pollack, 1974) W_i of $\vec{\phi}$ on M_i at p_i,

$$W_t = \frac{1}{2\pi} \int_{\partial \Sigma_i} d \arctan \left[\frac{\phi^2}{\phi^1} \right]$$

where $\partial\Sigma_i$ is the boundary of a neighborhood Σ_i of p_i on M_i with $p_i \notin \partial\Sigma_i$. The Winding numbers W_i correspond to the first homotopy group $\pi[S^1] = Z$ (the set of integers). By making use of (6), it can be precisely proved that

$$W_i = \frac{1}{2\pi} \int_{\partial \Sigma_i} n^*(\varepsilon_{ab} n^a dn^b) \tag{17}$$

where n^* is the pullback of the map n. This is another definition of W_i by the Gauss map (Dubrosin *et al.*, 1985) n: $\partial\Sigma_i \to S^1$. In topology this means that, when the point $v = (v^1, v^2)$ covers $\partial\Sigma_i$ once, the unit vector n^a will cover S^1 a total of W_i times, which is a topological invariant and is also called the degree of the Gauss map. Using the Stokes theorem in the exterior differential form and (17), one can deduce that

$$\begin{aligned} W_i &= \frac{1}{2\pi} \int_{\Sigma_i} \varepsilon_{ab}\, \partial_A n^a \partial_B n^b dv^A \wedge dv^B \\ &= \frac{1}{2\pi} \int_{\Sigma_i} \varepsilon^{AB}\, \varepsilon_{ab} \partial_A n^a \partial_B n^b d^2 v \end{aligned}$$

Then, by duplicating the above process, we have

$$W_i = \int_{\Sigma_i} \delta(\vec{\phi}) J \left(\frac{\phi}{v} \right) d^2 v \tag{18}$$

Substituting (16) into (18), and considering that only one $p_i \in \Sigma_i$, we get

$$\begin{aligned} W_i &= \int_{\Sigma_i} c_i \delta(N_i) J \left(\frac{\phi}{v} \right) d^2 v \\ &= \int_{\Sigma_i} \int_{N_i} c_i \frac{1}{\sqrt{g_x} \sqrt{g_v}} \delta^4 (\vec{x} - \vec{z}_i(u^1, u^2)) J \left(\frac{\phi}{v} \right) \sqrt{g_u} d^2 u \sqrt{g_v} d^2 v \end{aligned}$$

where $g_v = \det(g_{AB})$. Because $\sqrt{g_u}\sqrt{g_v} d^2 u\, d^2 v$ is the invariant volume element of the product manifold $M_i \times N_i$, so it can be rewritten as $\sqrt{g_x} d^4 x$. Thus, by calculating the integral and with positivity of c_i, we get

$$c_i = \frac{\beta_i \sqrt{g_v}}{|J(\phi/v)_{p_i}|} = \frac{\beta_i \eta_i \sqrt{g_v}}{J(\phi/v)_{p_i}} \tag{19}$$

where $\beta_i = |W_i|$ is a positive integer called the Hopf index (Milnor, 1965) of the ϕ-mapping on M_i; this means that when the point v covers the neighborhood of the zero point p_i once, the function $\bar{\phi}$ covers the corresponding region in $\bar{\phi}$-space β_i times, and $\eta_i = \text{sign}\, J(\phi/v)_{p_i} = \pm 1$ is the Brouwer degree (Milnor, 1965) of the ϕ-mapping. Substituting this expression for c_i and (16) in (12), we obtain the total expansion of the string current

$$j^{\mu\nu} = \frac{L_p}{\sqrt{g_x}} \sum_{i=1}^{l} \frac{\beta_i \eta_i \sqrt{g_v}}{J(\phi/v)|_{p_i}} \delta(N_i) J^{\mu\nu}\left(\frac{\phi}{x}\right)$$

From the above equation, we conclude that the inner structure of $j^{\mu\nu}$ is labeled by the total expansion of $\delta(\bar{\phi})$, which includes the topological information β_i and η_i.

With the discovery of an explicit four-particle amplitude that combines the narrow-resonance approximation with Regge behavior and crossing symmetry, some physicists began to study dual resonance models, i.e., string models, which can be generated from our topological tensor current theory. It is obvious that, in (13), when u^1 and u^2 are taken to be timelike evolution parameter and spacelike string parameter, respectively, the inner structure of $j^{\mu\nu}$ just represents l strings moving in the 4-dimensional Riemann–Cartan manifold X. The 2-dimensional singular submanifolds N_i ($i = 1, \ldots, l$) are their world sheets. Here we see that the strings are generated where $\bar{\phi} = 0$ and are not tied to any concrete models. Furthermore, we see that the Hopf indices β_i and Brouwer degrees η_i classify these strings. More precisely the Hopf indices β_i characterize the absolute values of the topological quantization; the Brouwer degrees $\eta_i = +1$ correspond to strings and $\eta_i = -1$ to antistrings.

4. THE EVOLUTION EQUATIONS OF STRINGS

First we give some useful relations to study many-string theory. On the ith singular submanifold N_i we have

$$\phi^a(x)|_{N_i} = \phi^a(z_i^1(u), \ldots, z_i^4(u)) \equiv 0$$

which leads to

$$\partial_\mu \phi^a \frac{\partial x^\mu}{\partial u^I}\bigg|_{N_i} = 0, \qquad I = 1, 2$$

Using this relation and the expression for the Jacobian matrix $J(\phi/v)$, we obtain

$$J^{\mu\nu}\left(\frac{\phi}{x}\right)\Bigg|_{\vec{\phi}=0} = \frac{1}{2}\,\varepsilon^{\mu\nu\lambda\rho}\,\varepsilon_{ab}\,\frac{\partial\phi^a}{\partial x^\lambda}\frac{\partial\phi^b}{\partial x^\rho}$$

$$= \frac{1}{2}\,\varepsilon^{\mu\nu\lambda\rho}\,\varepsilon_{ab}\,\frac{\partial\phi^a}{\partial v^A}\frac{\partial\phi^b}{\partial v^B}\frac{\partial v^A}{\partial x^\lambda}\frac{\partial v^B}{\partial x^\rho}$$

$$= \frac{1}{2}\,\varepsilon^{\mu\nu\lambda\rho}\,\varepsilon_{AB}\,J\left(\frac{\phi}{v}\right)\frac{\partial v^A}{\partial x^\lambda}\frac{\partial v^B}{\partial x^\rho} \qquad (20)$$

From this expression, the rank-two tensor current can be expressed by

$$j^{\mu\nu} = \frac{L_p}{2\sqrt{g_x}}\sum_{i=1}^{l}\beta_i\eta_i\sqrt{g_v}\,\delta\,(N_i)\varepsilon^{\mu\nu\lambda\rho}\varepsilon_{AB}\,\frac{\partial v^A}{\partial x^\lambda}\frac{\partial v^B}{\partial x^\rho} \qquad (21)$$

Corresponding to the rank-two topological tensor currents $j^{\mu\nu}$, it is easy to see that the multistring Lagrangian is

$$L = \frac{1}{L_p}\sqrt{\frac{1}{2}\,g_{\mu\lambda}g_{\nu\rho}\,j^{\mu\nu}j^{\lambda\rho}} = \delta(\vec{\phi})$$

which includes the total information of strings in X and is the generalization of Nielsen's Lagrangian for strings (Nielsen and Olesen, 1973). The action in X is expressed by

$$S = \int_X L\sqrt{g_x}\,d^4x = \sum_{i=1}^{l}\beta_i\eta_i\int_{N_i}\sqrt{g_u}\,d^2u = \sum_{i=1}^{l}\beta_i\eta_i S_i$$

where S_i is the area of the singular manifold N_i. The Nambu–Goto action (Nambu, 1970; Forster, 1974; Orland, 1994), which is the basis of many works on string theory, is derived naturally from our theory. From the principle of least action, we obtain the multistring evolution equations

$$g^{IJ}\,\frac{\partial g_{\nu\lambda}}{\partial x^\mu}\frac{\partial x^\nu}{\partial u^I}\frac{\partial x^\lambda}{\partial u^J} - 2\,\frac{1}{\sqrt{g_u}}\frac{\partial}{\partial u^I}\left(\sqrt{g_u}g^{IJ}\,g_{\mu\nu}\,\frac{\partial x^\nu}{\partial u^J}\right) = 0, \qquad I, J = 1,2 \qquad (22)$$

As a matter of fact, this is just the equation of a harmonic map (Duan *et al.*, 1992).

5. CONCLUSION

In summary, we have studied the topological quantization of strings in Riemann–Cartan space–time by making use of composed gauge theory and ϕ-mapping topological current theory. As a result, the strings are generated from the zero points of the ϕ-mapping and the topological quantum numbers of these strings are the winding numbers, which are determined by the Hopf indices and the Brouwer degrees of the ϕ-mapping; the singular manifolds of ϕ are just the evolution surfaces of these strings. The whole theory in this paper is not only covariant under general coordinate transformations, but also gauge invariant.

ACKNOWLEDGMENT

This work was supported by the National Natural Science Foundation of China.

REFERENCES

Bakas, I. (1993). *Phys. Lett. B* **319**, 457.
Bakas, I. (1994). *Int. J. Mod. Phys. A* **9**, 3443.
Bakas, I., and Sfetsos, K. (1996). *Phys. Rev. D* **54**, 3995.
Cartan, È. (1922). *C. R. Acad. Sci.* **174**, 593.
Cvetič, M., and Youm, D. (1996). *Phys. Rev. D* **54**, 2612.
De Sabbata, V. (1994). *Nuovo Cimento* **107A**, 363.
De Sabbata, V., *et. al.* (1991). *Int. J. Theor. Phys.* **30**, 167.
Duan, Y. S., and Lee, X. G. (1995). *Helv. Phys. Acta* **68**, 513.
Duan, Y. S., and Meng, X. H. (1993). *J. Math. Phys.* **34**, 1549.
Duan, Y. S., and Zhang, S. L. (1990). *Int. J. Eng. Sci.* **28**, 689.
Duan, Y. S., *et al.* (1992). *Gen. Rel. Grav.* **24**, 1033.
Duan, Y. S., *et al.* (1994). *J. Math. Phys.* **35**, 1.
Dubrosin, B. A., *et. al.* (1985). *Modern Geometry Method and Application*, Part II, Springer-Verlag New York.
Forster, D. (1974). *Nucl. Phys. B* **81**, 84.
Guillemin, V., and Pollack, A. (1974). *Differential Topology*, Prentice-Hall, Englewood Cliffs, New Jersey.
Hindmarsh, M. B., and Kibble, T. W. B. (1995). *Rep. Prog. Phys.* **58**, 477.
Jeannerot, R. (1996). *Phys. Rev. D* **53**, 5426.
Larsen, A. L., and Sánchez, N. (1996). *Phys. Rev. D* **54**, 2483, 2801.
Milnor, J. W. (1965). *Topology, From the Differential Viewpoint*, University Press of Virginia, Charlottesville.
Nambu, Y. (1970). *Lectures at the Copenhagen Symposium*.
Nash, S., and Sen, S. (1983). *Topology and Geometry for Physicists*, Academic Press, London.
Nielsen, H. B., and Olesen, P. (1973). *Nucl. Phys. B* **57**, 367.
Orland, P. (1994). *Nucl. Phys. B* **428**, 221.
Robinson, M., and Ziabicki, J. (1996). *Phys. Rev. D* **53**, 5924.
Rovelli, C. (1998). *Strings, loops and others: A critical survey of the present approaches to quantum gravity* (gr-qc/9803024).
Schouten, J. A. (1951). *Tensor Analysis for Physicists*, Clarendon Press, Oxford.
'tHooft, G. (1974). *Nucl. Phys. b* **79**, 276.